MA 3046 - Matrix Analysis

Problem Set 6 - Section IV - Systems of Equations

1. a. Approximately how many floating point operations (flops) will be required to solve a $25,000 \times 25,000$ linear system by Gaussian elimination?

b. Assuming one used a computer capable of executing a sustained 80 million floating point operations per second (80Mflops), approximately how long would it take to solve this system?

c. What would be the minimum available RAM required on this system in order not to have to page out to swap during execution of this solution?

2. A particular PC, using Gaussian elimination, solves a system of $1,000$ equations in $1,000$ unknowns in about two seconds, with no apparent signs of excessive paging or swapping. Approximately how long would it this same PC to solve a $3,000 \times 3,000$ system, assuming neither paging nor swapping problems arise?

3. Consider the following system:

$$\begin{array}{rcrcrcr} 2.25\, x_1 & - & 3.31\ \ x_2 & + & 1.28\, x_3 & = & 3.43 \\ 7.88\, x_1 & - & 11.5\ \ \ x_2 & + & 7.15\, x_3 & = & 6.35 \\ 4.23\, x_1 & + & 3.70\ \ x_2 & + & 2.85\, x_3 & = & 6.39 \end{array}$$

a. Simulate the solution of this system by Gaussian elimination without partial pivoting in a three-digit, decimal machine with rounding of all intermediate results.

b. Repeat the solution of part a., but this time simulate Gaussian elimination with partial pivoting in the same machine.

c. Repeat the solution of part a., but this time simulate Gaussian elimination with full pivoting in the same machine.

d. Compare your solutions from parts a., b. and c. above to the true (full MATLAB precision) solution.

e. Calculate the residuals for each of the solutions from parts a. and b. above. What can you conclude about the accuracy of the solutions from the size of their respective residuals?

4. Consider the following system:

$$\begin{array}{rcrcrcl} 4.55\ x_1 & + & 2.39\ x_2 & + & 4.18\ x_3 & = & 7.52 \\ 2.40\ x_1 & + & 2.04\ x_2 & + & 3.75\ x_3 & = & 4.77 \\ 3.19\ x_1 & - & 2.06\ x_2 & - & 4.23\ x_3 & = & 1.45 \end{array}$$

a. Simulate the solution of this system by Gaussian elimination without partial pivoting in a three-digit, decimal machine with rounding of all intermediate results.

b. Repeat the solution of part a., but this time simulate Gaussian elimination with partial pivoting in the same machine.

c. Compare your solutions from parts a. and b. above to the true (full MATLAB precision) solution.

d. Calculate the residuals for each of the solutions from parts a. and b. above. What can you conclude about the accuracy of the solutions from the size of their respective residuals?

5. Calculate the true (MATLAB) solution for the system:

$$\begin{array}{rcrcrcl} 4.55\ x_1 & + & 2.39\ x_2 & + & 4.18\ x_3 & = & 7.56 \\ 2.40\ x_1 & + & 2.04\ x_2 & + & 3.75\ x_3 & = & 4.72 \\ 3.19\ x_1 & - & 2.06\ x_2 & - & 4.23\ x_3 & = & 1.50 \end{array}$$

(Notice the left-hand side here is identical to, and the right-hand side only a relatively small perturbation of problem 4.) Compare the exact (MATLAB) solutions to both problems. What does that comparison suggest about the condition of this matrix?

6. Consider the matrices:

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{bmatrix} 3 & -1 & 2 & 4 \\ -4 & 2 & -3 & -3 \\ 2 & -2 & 5 & 2 \\ 3 & -3 & 1 & -1 \end{bmatrix}$$

a. Show that $\mathbf{P}$ is a permutation matrix by showing that it can be created as a product of elementary permutation matrices.

b. Show that $\mathbf{P}^{(i)}\,\mathbf{P}^{(i)} = \mathbf{I}$, where $\mathbf{P}^{(i)}$ denotes any one of the elementary permutation matrices you found in part a., and therefore that ${\mathbf{P}^{(i)}}^{-1} = \mathbf{P}^{(i)}$. Why should this be true in general for *elementary* permutations? Why, however, should $\mathbf{P}^{-1} = \mathbf{P}$ **not** be true for more general permutations, e.g. why does $\mathbf{P}\,\mathbf{P} \neq \mathbf{I}$ here?

c. Compute $\mathbf{P}\,\mathbf{A}$ and $\mathbf{A}\,\mathbf{P}$ and show that the results are as predicted by theory.

7. Solve the following system by $\mathbf{L}\,\mathbf{U}$ Decomposition with Partial Pivoting, and forward/backward substitution:

$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & 2x_3 & = & 1 \\ 6x_1 & - & 2x_2 & + & 5x_3 & = & -1 \\ -3x_1 & + & 2x_2 & - & x_3 & = & -5 \end{array}$$

8. Solve the following system by $\mathbf{L}\,\mathbf{U}$ Decomposition with Partial Pivoting, and forward/backward substitution:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 2x_3 & - & 2x_4 & = & 2 \\ 3x_1 & + & x_2 & + & 6x_3 & - & 3x_4 & = & -6 \\ -2x_1 & + & x_2 & + & x_3 & + & 2x_4 & = & -1 \\ -x_1 & + & x_2 & - & 8x_3 & + & 3x_4 & = & 10 \end{array}$$

9. a. Show that for any invertible matrices $\mathbf{A}$ and $\mathbf{C}$ in $\mathbb{C}^{m\times m}$,

$$\begin{bmatrix} \mathbf{A} & \vdots & \mathbf{B} \\ \cdot & \cdots & \cdot \\ \mathbf{0} & \vdots & \mathbf{C} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \vdots & -\mathbf{A}^{-1}\mathbf{B}\mathbf{C}^{-1} \\ \cdots & \cdots & \cdots\cdots\cdots \\ \mathbf{0} & \vdots & \mathbf{C}^{-1} \end{bmatrix}$$

(Note that from this we can immediately conclude that the inverse of any upper triangular matrix $\mathbf{U}$ is also upper triangular, provided all the diagonal elements of $\mathbf{U}$ are non-zero.)

b. Using the result from part a. above, show that, if $\mathbf{U}$ is invertible, then the diagonal elements of $\mathbf{U}^{-1}$ are precisely the reciprocals of the diagonal elements of $\mathbf{U}$.

c. Based on your result from part b., show that the condition number of an upper triangular matrix $\mathbf{U}$ satisfies

$$\kappa(U) \geq \frac{\max_{i,j} |u_{ij}|}{\max_i |u_{ii}|}$$

This result, of course, explains why either a large growth factor (ρ) or the appearance of unavoidable small pivots is a sure sign of trouble in Gaussian elimination.

10. Consider the system of linear equations:

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} 3.44 & 12.4 \\ -0.345 & -1.32 \end{bmatrix} = \begin{bmatrix} 1.00 & 0 \\ -.100 & 1.00 \end{bmatrix} \begin{bmatrix} 3.44 & 12.4 \\ 0 & -0.0800 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 1.14 \\ -0.620 \end{bmatrix} \quad .$$

In a computer for which (normal) single precision is three decimal digits, with rounding of all intermediate terms, a computed solution to this problem is

$$\tilde{\mathbf{x}}^{(0)} = \begin{bmatrix} -22.5 \\ 6.33 \end{bmatrix} \quad .$$

a. Perform a single iteration of iterative improvement on the above solution.

b. Based on your answer to part a, do you feel this matrix is ill-conditioned in a three decimal digit machine. (*Justify your answer.*)

11. Find the Cholesky factorization of the symmetric, positive definite matrix

$$\mathbf{A} = \begin{bmatrix} 16 & -4 & -12 \\ -4 & 82 & -24 \\ -12 & -24 & 22 \end{bmatrix}$$